1. Directed Graph.

A graph $G = (V, E)$ with a set of vertices V and a set of edges E.

$$V = V(G), \quad E = E(G) = V \times V$$

An edge pointing from node i to node j is denoted as (i, j) or $i \to j$

Graph G is said to be
- unidirected if $(i,j) \in E \Rightarrow (j,i) \notin E$
- bidirected if $(i,j) \in E \Rightarrow (j,i) \in E$
- undirected if $E = \{\{i,j\} : (i,j) \in E\}$ (usually uses U rather than G)

For undirected graph U, i & j are adjacent if $\{i,j\} \in E(U)$

For bidirected graph G, it's associated undirected graph is is denoted to be $U(G)$.

The <u>transpose graph G^T</u> of G is defined by.

1) $V(G^T) = V(G)$: same nodes
2) $\forall (i,j) \in E(G) \Rightarrow (j,i) \in E(G^T)$: switch (i,j).

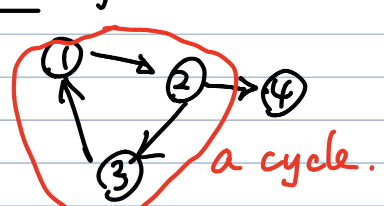
Corollary: $G^T = G$ iff G is bidirected.

$E(G^T) \cap E(G) = \emptyset$ iff G is unidirected.

A graph G' is a <u>subgraph</u> of G if

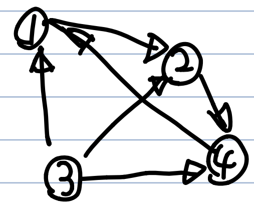
$$\begin{cases} V(G') \subseteq V(G) \\ E(G') \subseteq E(G) \cap (V(G') \times V(G')) \end{cases}$$

A graph is called <u>acyclic</u> if there is no cycle that $i_1 \to i_2 \to \cdots i_k \to i_1$ for $i_j \in V(G)$.



A tournament is a unidirected graph in which every pair of distinct vertices is connected by a single directed edge.

E.g.



A subgraph G' is said to be induced by $J \in V(G)$ if

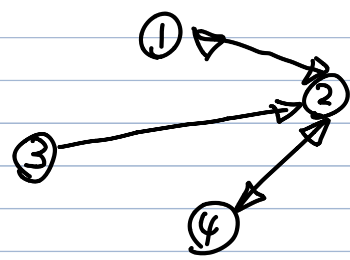
$$V(G') = J \quad E(G') = J \times J.$$

We use notation $G' = G|_J$. We sometimes call G' to be a projection of G onto J.

Lemma 1. Every tournament on n vertices includes an acyclic subgraph induced on $1 + \lfloor \log_2 n \rfloor$ vertices. Stearns, Erdös and Moser.

An independent set of a graph G is a set of vertices with no edge among them. The independence number $\alpha(G)$ is the size of the largest independent set of graph G.

E.g.



$\alpha(G) = 3$, the independence set is $\{1, 3, 4\}$.

A clique K of graph G is a set of vertices such that there is an edge from every vertex in K to every other vertex in K. The clique number $\omega(G)$ is the size of the largest clique of graph G.

The complement/inverse of Graph G is a graph $\bar{G}$ such that

$$V(G) = V(\bar{G}), \quad E(G) \cap E(\bar{G}) = \emptyset, \quad E(G) \cup E(\bar{G}) = V(G) \times V(G).$$

With G & $\bar{G}$, we state that

$$\omega(G) = \alpha(\bar{G}).$$

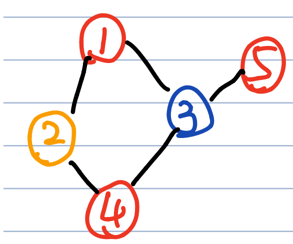
The above equality holds even for undirected graphs.

2. Coloring the Graph. (Undirected Graph)

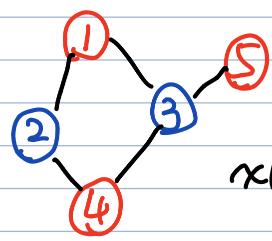
A coloring of an undirected graph U is a mapping that assigns a color to each vertex such that no two adjacent vertices share the same color.

The chromatic number $\chi(U)$ is the minimum number of colors such that a coloring of graph U exists.

A b-fold coloring assigns a set of b colors to each vertex such that no two vertices share some color in common, $\chi^{(b)}(U)$ is the minimum number of colors.

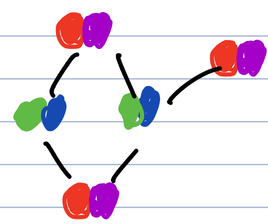


3 colors



2 colors

$\chi(U) = 2.$



2-fold coloring.

$\chi^{(2)}(U) = 4$

The fractional chromatic number of the graph is defined as

$$\chi_f(U) = \inf_b \frac{\chi^{(b)}(U)}{b} = \lim_{b\to\infty} \frac{\chi^{(b)}(U)}{b},$$

the limit exists by Fekete's Lamma since $\chi^{(b)}(U)$ satisfies subadditive,

$$\chi^{(a+b)}(U) \le \chi^{(a)}(U) + \chi^{(b)}(U).$$

So $\frac{\chi^{(b)}(U)}{b}$ is a non-increasing function, Therefore

$$\chi_f(U) \le \chi(U)$$

For any coloring, vertices of the same color form an independent set. Let I be a set of all independent sets in $\mathcal{U}$, we use the following optimization problem to characterize the chromatic number and the fractional chromatic number.

$$\underset{\gamma_J,\, I,}{\text{minimize.}} \sum_{J \in I} \gamma_J$$

$$\text{s.t.} \quad \sum_{\substack{J \text{ such that} \\ i \in J}} \gamma_J \geq 1, \text{ for each } i \in V(\mathcal{U})$$

1) a vertex can be in multiple independent sets when $b > 1$. (b-fold)

2). If we require $\gamma_J \in \{0, 1\}$, the optimal objective function value is the chromatic number. If we require (relax) $\gamma_J \in [0, 1]$, the objective value is the fractional chromatic number.

Why? If we require $\gamma_J \in \{0, 1\}$, the optimal result γ_J^*, I^* represents the optimal partition to assign colors to each vertex. The objective value equals the number of colors; the constraint is to make sure that vertex i showed up in at least 1 independent set.

If we let $\gamma_b \in [0, 1]$, we can multiply each term by b and let $\gamma_J' = b\,\gamma_J$ s.t $\gamma_J' \in \{0, 1\}$,

$$\min \sum_{J \in I} \gamma_J'$$

$$\text{s.t.} \quad \sum_{\substack{J \text{ s.t} \\ i \in J}} \gamma_J' \geq b \quad \text{for each } i.$$

$$b > 0.$$

With the explanation for the case where $\gamma_J \in \{0, 1\}$, the new problem is a b-fold coloring problem. Since $\gamma_J \in [0, 1]$ is a relaxed constraint, we can achieve the following inequality.

$$\chi_f(U) \leq \chi(U).$$

Under any circumstance, vertices in a clique are colored distinctly, so the minimum # of colors must be no less than $\omega(U)$. We have

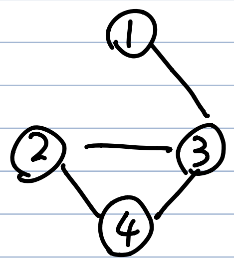
$$\underline{\omega(U) \leq \chi_f(U) \leq \chi(U)}$$

Scheinerman & Ullman state that,

$$\chi_f(U) \geq \frac{|V(U)|}{\alpha(U)} = \frac{|V(\bar{U})|}{\omega(\bar{U})}$$

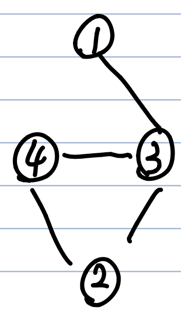
An <u>automorphism</u> of an undirected graph U is a bijective function $\pi: V(U) \to V(U)$ such that for any two vertices $i, j \in V(U)$, $\pi(i), \pi(j)$ are adjacent iff i and j are adjacent.

An undirected graph U is said to be <u>vertex transitive</u> if for any two vertices i, j, there exists an automorphism π s.t $\pi(i) = j$



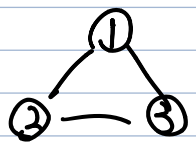
U

π:
$1 \to 1$
$2 \to 4$
$3 \to 3$
$4 \to 2$



U'

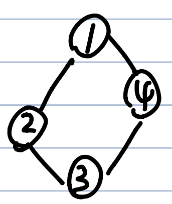
← not a vertex transitive graph, because it's impossible to find a π s.t $\pi(1) = 3$



π_1: $1 \to 2$, $2 \to 3$, $3 \to 1$

π_2: $1 \to 3$, $2 \to 1$, $3 \to 2$

. . ← A connected graph is a simple example of vertex transitive graph.



π:
$1 \to 3$
$2 \to 2$
$3 \to 1$
$4 \to 4$

← also a vertex transitive graph

Vertex transitive graphs are highly symmetric, because without knowing the index of each vertex, standing at any vertex, the graph look the same

For vertex transitive graphs. $\chi_f(U) = \frac{|V(U)|}{\alpha(U)}$.

Four Color Map Theorem.

For loopless planar G, the chromatic number of it's dual graph is $\chi(G^*) \leq 4$.

- Dual graph is a projected graph of plannar G s.t. each area i maps to a vertex in its dual graph. $\pi(i)$. $\pi(i)$ and $\pi(j)$ are adjacent iff i and j are adjacent.
- Planar graph is a graph that can be embedded in the 2-D plane such that edges intersect only at their end points. i.e. no two edges cross each other.

3. Perfect Graph. (undirected graph)

An undirected graph is perfect if for every induced subgraph $U|_J$, the clique number equals the chromatic number, i.e. $\omega(U|_J) = \chi(U|_J)$.

Proposition 1. (Chudnovsky, Robertson, Seymour and Thomas).

An undirect graph is perfect iff no induced subgraph is an odd cycle of length at least five or the complement of one.

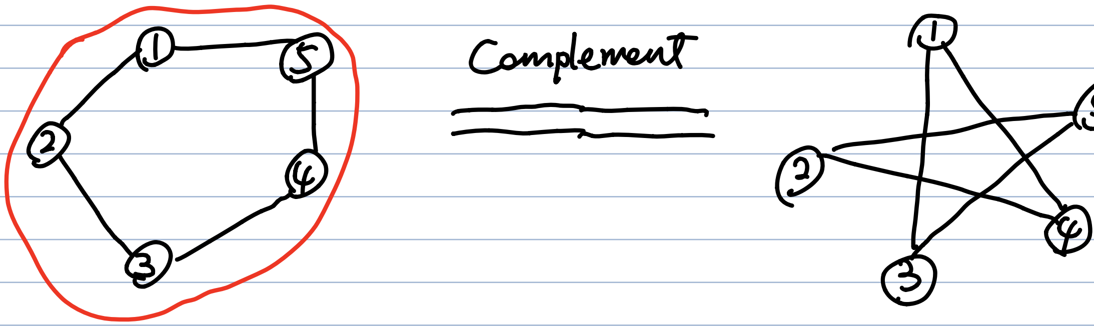
(odd hole) (antihole)

Perfect graph theorem

A graph is perfect iff it's complement is perfect.

Strong perfect graph theorem (same as proposition 1)

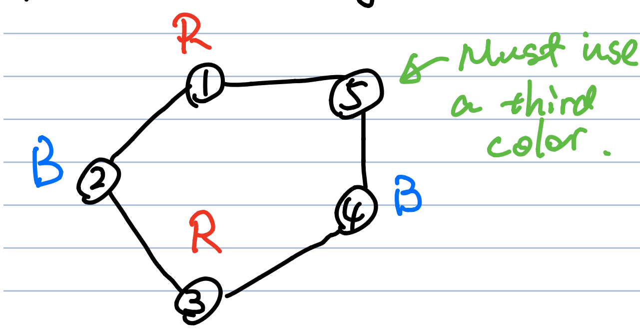
Perfect graph are graphs G where neither G nor $\bar{G}$ contains an induced cycle of odd length no less than 5.



For an odd cycle of length ≥ 5, we cannot use 2 colors to color the cycle, but the clique number is 2.

But for an odd cycle of length 3, the clique number $= 3$.

For an even cycle, we can always use two colors



Try to color

Let $\mathcal{U}$ be an undirected graph on n vertices. For each clique k in $\mathcal{U}$, the <u>incidence vector</u> $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]$ is defined by.

$$x_i(k) = \mathbb{1}(\text{vertex } i \in k).$$

E.g. if K has 3 vertices $1, 3, 4$, and $n = 5$, the incidence vector is

$$[1 \ 0 \ 1 \ 1 \ 0]$$

Let K be a set of all cliques in $\mathcal{U}$, <u>the clique polytope</u> of $\mathcal{U}$ is defined by the convex hull of the incidence vectors of cliques of $\mathcal{U}$ as

$$P^*(\mathcal{U}) = \left\{ \sum_{k \in K} \delta_k \cdot x(k) : \delta_k \geq 0, \sum_{k \in K} \delta_k = 1 \right\}$$

The convex polytope associated with $\mathcal{U}$ is defined as

$$P(\mathcal{U}) = \left\{ [x_1, x_2, \cdots, x_n] : \sum_{i \in I} x_i \leq 1 \text{ for all independent sets } I, \ x_i \geq 0 \right\}$$

Recall that independent set I is a set of vertices with the same color.

It's not very hard to believe that $P^*(U) \subseteq P(U)$, because $P^*(U)$ is defined on cliques, which must be neighboring vertices while $P(U)$ is defined on colorings, where vertices can be far away. And clique is a strict constraint than coloring.

Lovász's perfect graph theorem

For any graph U, the following statements are equivalent.

1) U is a perfect graph.
2) $P^*(U) = P(U)$
3) $\bar{U}$ is perfect.

4. Graph Products.

We use $i \sim j$ denotes that vertex i & j are adjacent.

1) Disjuctive Product ($U_1 \vee U_2$)

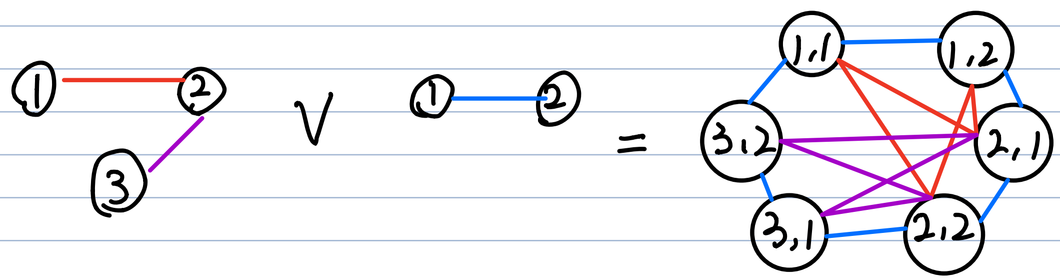
Given two undirected graphs U_1 and U_2, the disjuctive product $U = U_1 \vee U_2$ is defined as.

$$V(U) = V(U_1) \times V(U_2) \quad \text{cartesion product.}$$

From U_1 From U_2

Vertex in U is denoted as a two-element tuple (i_1, i_2)

$$\underbrace{\text{Vertex } (i_1, i_2) \sim (j_1, j_2)}_{\text{in } U} \quad \text{iff} \quad \underbrace{i_1 \sim j_1}_{\text{in } U_1} \ \boxed{\text{or}} \ \underbrace{i_2 \sim j_2}_{\text{in } U_2}$$



$U^{\vee k}$ is a graph generated by producting U for k times.

For chromatic number, $\chi(U_1 \vee U_2) \le \chi(U_1)\,\chi(U_2)$.

Lemma (Scheinerman & Ullman)

$$\chi_f(U_1 \vee U_2) = \chi_f(U_1)\,\chi_f(U_2)$$

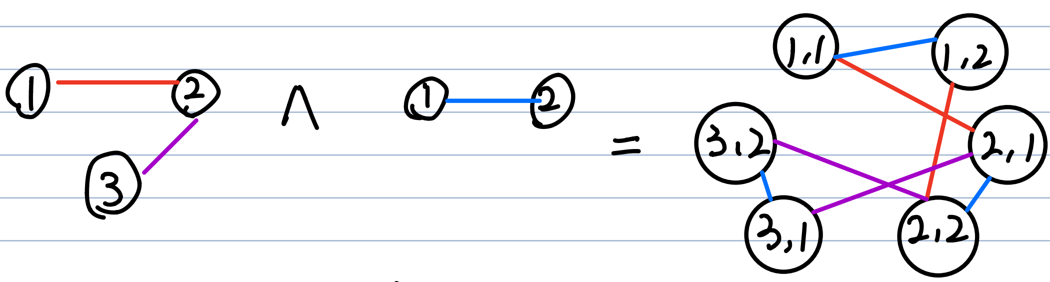
$$\chi_f(U) = \lim_{k\to\infty} \sqrt[k]{\chi(U^{\vee k})} = \inf_k \sqrt[k]{\chi(U^{\vee k})}$$

2) Cartesion Product ($U_1 \wedge U_2$)

Given two undirected graphs U_1 & U_2, the Cartesion Product $U = U_1 \wedge U_2$ is defined by

$$V(U) = V(U_1) \times V(U_2) \quad \text{(same as disjuctive product)}$$

$$(i_1, i_2) \sim (j_1, j_2) \text{ iff } \begin{cases} (i_1 = j_1,\ i_2 \sim j_2); \\ \quad\text{or} \\ (i_1 \sim j_1,\ i_2 = j_2); \end{cases}$$

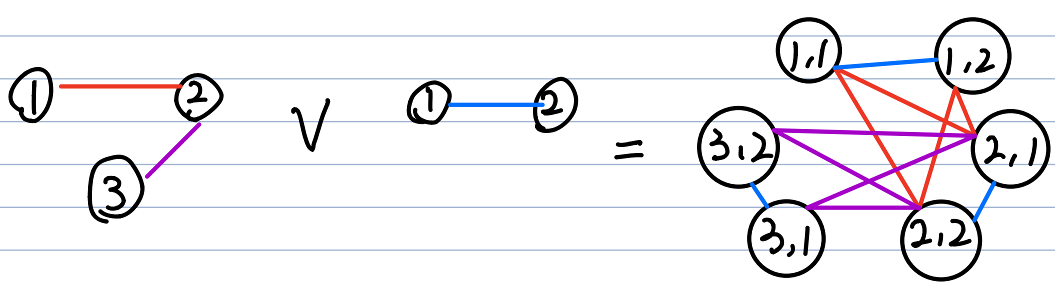


Lemma (Sabidussi)

$$\chi(U_1 \wedge U_2) = \max(\chi(U_1), \chi(U_2))$$

3) Lexicographic Product ($U_1 \circ U_2$)

$U = U_1 \circ U_2$ with $(i_1, i_2) \sim (j_1, j_2)$ iff $\begin{cases} i_1 \sim j_1 \\ \quad\text{or} \\ i_1 = j_1,\ i_2 \sim j_2 \end{cases}$



"Lexicographic" indicates this product is not commutative, i.e

$U_1 \circ U_2 \neq U_2 \circ U_1$

But $\chi_f(U_1 \circ U_2) = \chi_f(U_2 \circ U_1) = \chi_f(U_1)\, \chi_f(U_2)$